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DIFFRACTION BY A WEDGE WITH A CIRCULAR CAP

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A solution of the problem of acoustic wave diffraction by a wedge with a circular cap, whose center is at the wedge apex, is constructed by the Fourier method. Asymptotics of the wave field for different relationships between the cap radius and the wavelength are obtained from the exact solution. The connection between the asymptotic behavior with respect to distance and the long- and shortwave asymptotics is discussed.

The two-dimensional problem of diffraction from a point source (at the point $M_0 = (r_0, \varphi_0))$ in a system consisting of an angle $0 < \varphi < 2\pi - \Phi$; $0 < \Phi \leqslant 2\pi$ and a circle $r = a$ is considered (see Fig. 1). A function u is investigated such that

$$(\Delta + k^2)\, u = \delta\,(M - M_0), \quad r > a, \quad 0 < \varphi < \Phi \tag{1}$$

$$\partial u/\partial r - iku = 0\,(r^{-1/2}), \quad r \to \infty$$

$$u\,|_S = 0, \quad S = S_0 \cup S_\Phi \cup S_a \tag{2}$$

$$S_0 = \{r > a,\ \varphi = 0\},\ S_\Phi = \{r > a,\ \varphi = \Phi\},\ S_a = \{r = a,\ 0 < \varphi < \Phi\}$$

and $|\operatorname{grad} u|^2$ is locally summable.

A unique solution of this problem is constructed by separation of variables

$$u = i\zeta \sum_{m=1}^{\infty} \left[J_{m\zeta}(\rho_*) - \frac{J_{m\zeta}(\alpha)}{H_{m\zeta}^{(1)}(\alpha)}\, H_{m\zeta}^{(1)}(\rho_*) \right] H_{m\zeta}^{(1)}(\rho_{**}) \sin m\zeta\varphi \sin m\zeta\varphi_0 \tag{3}$$

$$\alpha = ka, \quad \rho_{**} = \max(\rho, \rho_0), \quad \rho_* = \min(\rho, \rho_0), \quad \rho = kr, \quad \rho_0 = kr_0, \quad \zeta = \frac{\pi}{\Phi}$$

where J, $H^{(1)}$ are the Bessel and Hankel functions of the first kind, respectively.

Let us investigate the asymptotics of the function u as $\rho_* \to \infty$ (and, therefore, $\rho_{**} \to \infty$) and different (α).

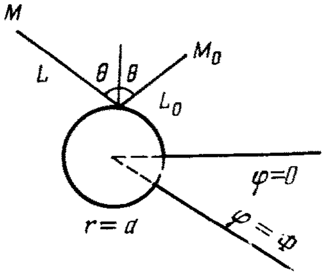

Fig. 1

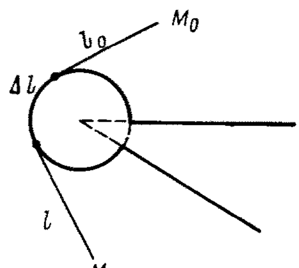

Fig. 2

1°. Let us start with the case $\alpha \to \infty$ when the radius of the cap significantly exceeds the wavelength. Let us first assume that $\rho_{**} \to \infty$). The solution is investigated by the usual methods of analyzing the shortwave asymptotics. (*)

*) Babich, V. M., Buldyrev, V. S., and Molotkov, I. A., Some mathematical methods used in diffraction theory. First All-Union School Seminar on Diffraction and Wave propagation, (Palanga, 1965). Moscow Khar'kov, 1968.

The function (3) is converted into a contour integral, whose asymptotic behavior depends essentially on the mutual location of the source M_0 and the observer. Omitting the long, but standard, calculations, let us present the results for two characteristic cases.

A. In the domain where there is just one wave reflected from the cap (see Fig. 1)

$$u = k^{-1/2}\left(L + L_0 + \frac{2LL_0}{a\cos\theta}\right)^{-1/2}\exp\left[ik\left(L+L_0\right)\right]\left(1+O\left(\frac{1}{\alpha}\right)\right) \tag{4}$$

(L_0 and L are members of the reflected ray, θ is the angle of reflection). Under the condition $\alpha^2/\rho_* \to 0$ the expression (4) corresponds to a cylindrical wave diverging from the apex of the angle

$$u = \left(\frac{\alpha}{2}\right)^{1/2}\cos^{1/2}\frac{\varphi-\varphi_0}{2}\,\frac{\exp\left[i\left(\rho+\rho_0\right)\right]}{\left(\rho\rho_0\right)^{1/2}}\left(1+O\left(\frac{1}{\alpha}\right)\right)\left(1+O\left(\frac{\alpha^2}{\rho_*}\right)\right) \tag{5}$$

B. If the point M is in the zone of geometric shadow (Fig. 2), then

$$u = A\alpha^{-2/3}\frac{a}{\left(ll_0\right)^{1/2}}\exp\left[iB\alpha^{1/3}\frac{\Delta l}{a}\right]\times\exp\left[ik\left(l+l_0+\Delta l\right)\right]\left(1+O\left(\alpha^{-2/3}\right)\right) \tag{6}$$

Here

$$A = 2^{-1/3}\left(w_1'\left(t_1\right)\right)^{-2},\quad B = 2^{-1/3}t_1$$

w_1 is the Airy-Fock function, t_1 is its root closest to the real axis, l and l_0 are the lengths of the tangents drawn from M and M_0 to the circle $r = a$, and Δl is the arc of wave slip. In obtaining (6) it can be noted that it is admissable to let ρ_*/α^2 tend to infinity. The field shadow zone also acquires the character of a cylindrical wave

$$u = A\alpha^{1/3}\exp\left[i\left(B\alpha^{1/3}+\alpha\left|\varphi-\varphi_0\right|\right)\right]\frac{\exp\left[i\left(\rho+\rho_0\right)\right]}{\left(\rho\rho_0\right)^{1/2}}\left(1+O\left(\alpha^{-2/3}\right)\right) \tag{7}$$
$$\times\left(1+O\left(\alpha^2/\rho_*\right)\right)$$

however, the amplitude and directivity pattern is completely different than in the exposed region.

The expressions (5) and (7) are not similar to the known edge wave from the apex of an angle without a cap (see [1], for instance), and tend to zero as α formally approaches zero. The formulas of geometric diffraction theory (4), (6), which are obtained for $\alpha \to \infty$, do not allow a passage to the limit for small α.

The result turns out to be analogous for other mutual locations of M and M_0

2°. Let $\alpha = O(1)$, i.e., the radius of the cap is commensurate with the wavelength. Now, in contrast to the preceding case, it is expedient to consider the cap a perturbation and to partition the solution into the sum

$$u = V + W \tag{8}$$

of the Green's function W of the exterior of the angle $0 < \varphi < 2\pi - \Phi$ with the Dirichlet boundary condition $W|_{\varphi=0} = W|_{\varphi=\Phi} = 0$, and the addition V due to the cap

$$V = i\zeta\sum_{m=1}^{\infty}\frac{J_{m\zeta}(\alpha)}{H_{m\zeta}^{(1)}(\alpha)}H_{m\zeta}^{(1)}(\rho)\,H_{m\zeta}^{(1)}(\rho_0)\sin m\zeta\varphi\sin m\zeta\varphi_0 \tag{9}$$

The function W has been studied well (see [1], for example). Let us examine just the addition V dependent on α. Values of $J_{m\zeta}(\alpha)$ decrease exponentially while $H^{(1)}_{m\zeta}(\alpha)$ grows with the increase in m when $m\zeta > \alpha$. Hence, only a small number of components on the order of $O(\alpha)$ introduces a contribution to the sum (9) for bounded α. The arguments ρ and ρ_0 in the numerator of the Hankel function are large compared to the subscript for each. Using the asymptotics

$$H^{(1)}_{\nu}(z) = 2^{1/2}(\pi z)^{-1/2}\exp\left[i\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right]\left(1 + O\left(\frac{\nu^2}{z}\right)\right), \quad \frac{\nu^2}{z} \to 0$$

and then adding the exponentially small components, we obtain

$$V \sim \frac{2}{\Phi}\,\frac{\exp[i(\rho+\rho_0)]}{(\rho\rho_0)^{1/2}}\sum_{m=1}^{\infty}\frac{J_{m\zeta}(\alpha)}{H^{(1)}_{m\zeta}(\alpha)}\exp(-im\zeta)\sin m\zeta\varphi\sin m\zeta\varphi_0 \tag{10}$$

The corrections for the cap corresponds in the far zone to a cylindrical wave of the same order in ρ and ρ_0 as for the edge wave from the apex of the angle, but with another pattern complexly dependent on α.

An ever smaller number of members of the series (9) (just as its asymptotics (10) in ρ_*) plays a noticeable part as α diminishes.

3°. In the long wavelength case $\alpha \to 0$ (the wavelength noticeably exceeds the radius of the cap), the series (9) and (10) are asymptotic in α and

$$\begin{aligned} V &\sim \frac{\pi^2}{\Gamma^2(\zeta)}\left(\frac{\alpha}{2}\right)^{2\zeta} H^{(1)}_{\zeta}(\rho)\,H^{(1)}_{\zeta}(\rho_0)\sin\zeta\varphi\sin\zeta\varphi_0(1 + O(\alpha^{2\zeta})) \\ &= -2\,\frac{\exp(-i\zeta)}{\Gamma^2(\zeta)}\left(\frac{\alpha}{2}\right)^{2\zeta}\frac{\exp[i(\rho+\rho_0)]}{(\rho\rho_0)^{1/2}}\times \\ &\quad \sin\zeta\varphi\sin\zeta\varphi_0(1 + O(\alpha^{2\zeta}))\left(1 + O\left(\frac{1}{\rho_*}\right)\right) \end{aligned} \tag{11}$$

under the condition that $\zeta \neq 1$. For $\zeta = 1$, the error $1 + O(\alpha^{2\zeta})$ in (11) is replaced by $1 + O(\alpha^2 / \ln\alpha)$. The form of the subsequent terms of the expansion V as $\alpha \to \infty$ depends on whether there is at least one integer among the numbers $m\zeta$. If so, then the asymptotic series contains powers and logarithms of α, otherwise, only powers. In particular the logarithmic terms are characteristic for those Φ for which there is no edge wave from the apex of the angle without a cap ($\alpha = 0$).

Upon replacement of the Dirichlet boundary condition (2) by the Neumann condition

$$\left.\frac{\partial u}{\partial r}\right|_{S_a} = \left.\frac{\partial u}{\partial\varphi}\right|_{S_\Phi} = \left.\frac{\partial u}{\partial\varphi}\right|_{S_0} \tag{12}$$

the correction for the influence of the cap is converted into

$$\begin{aligned} V^{(N)} &= i\zeta\sum_{m=0}^{\infty}\delta_m\frac{J_{m\zeta}(\alpha)}{H^{1}_{m\zeta}(\alpha)}H^{(1)}_{m\zeta}(\rho)\,H^{(1)}_{m\zeta}(\rho_0)\cos m\zeta\varphi\cos m\zeta\varphi_0 \\ \delta_0 &= 1; \quad \delta_m = {}^1/_2, \quad m \neq 1 \end{aligned} \tag{13}$$

The function (13) evidently is on the order of $O(1/\ln\alpha)$ as $\alpha \to 0$. If the Neumann condition is imposed on the faces of the angle S_0 and S_Φ and the Dirichlet condition on the cap S_a, then $V = O(\alpha)$.

4°. A more detailed analysis of the derivation of (10) shows that it is also valid as $\alpha \to \infty$ if only $\alpha^2 / \rho_* \to 0$. It is easy to study the sum in (10) which now contains the large diameter α by known methods. (*).

In the lighted region (see Fig. 1), the right side of (10) is converted into a contour integral. Evaluating its asymptotic behavior by the saddle-point method, the principal member in (5) can be obtained. In this case the function W has a lower order: $W = O(\rho_*^{-1/2} V)$.

In the shadow, the partition (8) is not expedient. An expression analogous to (10) can be obtained for the complete field $u = V + W$, and it can be converted into a contour integral. The asymptotics of this latter is given by the residue and agrees with (7).

Thus, the formulas obtained for the scattering problem: $\alpha = O(1)$, $\rho_* \to \infty$, have a common domain of applicability with the shortwave and longwave asymptotics.

Let us note that a problem which converges to that considered when $\Phi = 2\pi$, was studied in [2]. Simple formulas have been obtained only for the shortwave case $\alpha \to \infty$, $\alpha \sim \rho \sim \rho_0$.

*) See the previous footnote 1.

REFERENCES

1. Borovikov, V. A., Diffraction by Polygons and Polyhedra. Nauka, Moscow, 1966.

2. Keller, J.B. and Magiros, D. G., Diffraction by a semi-finite screen with a rounded end. Communs. Pure and Appl. Math., Vol. 14, No.3, 1961.

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